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# Galilei-invariant nonlinear systems of evolution equations

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**Abstract.** All systems of $(n+1)$-dimensional quasilinear second-order evolution equations invariant under chain of algebras $AG(1.n) \subset AG_1(1.n) \subset AG_2(1.n)$ are described. The results obtained are illustrated by the examples of the nonlinear Schrödinger equations, Hamilton–Jacobi-type systems and reaction–diffusion equations.

## 1. Introduction

The $(n+1)$-dimensional diffusion (heat) system of equations

$$\begin{aligned} \lambda_1 U_t &= \Delta U \\ \lambda_2 V_t &= \Delta V \end{aligned} \tag{1}$$

where $U = U(t,x), V = V(t,x)$ are unknown differentiable real functions, $U_t = \partial U/\partial t, V_t = \partial V/\partial t, x = (x_1, \ldots, x_n), \lambda_1, \lambda_2 \in \mathbb{R}$, is known to be invariant under the generalized Galilei algebra $AG_2(1.n)$ [1,2]

$$P_t = \partial_t \qquad P_a = \partial_a \tag{2a}$$

$$Q_\lambda = \lambda_1 U \partial_U + \lambda_2 V \partial_V \qquad G_a = tP_a - \frac{x_a}{2} Q_\lambda \qquad J_{ab} = x_a P_b - x_b P_a \tag{2b}$$

$$D = 2tP_t + x_a P_a + I_\alpha \tag{2c}$$

$$\Pi = t^2 P_t + t x_a P_a - \tfrac{1}{4}|x|^2 Q_\lambda + t I_\alpha, \alpha_k = -\tfrac{1}{2}n. \tag{2d}$$

In relations (2) and elsewhere hereinafter $I_\alpha = \alpha_1 U \partial_U + \alpha_2 V \partial_V$, $\partial_U \equiv \partial/\partial U$, $\partial_V \equiv \partial/\partial V$, $\partial_t \equiv \partial/\partial t$, $\partial_a \equiv \partial/\partial x_a$, $\alpha_k \in \mathbb{R}$, $k = 1, 2$ and a summation is assumed from 1 to $n$ over repeated indices.

The algebra produced by the operators $(2a), (2b)$ is called the Galilei algebra $AG(1.n)$, and its extension by using the operator $(2c)$ will be refereed to as $AG_1(1.n)$ [1,2].

Clearly, the unit operators $I_\alpha$ and $Q_\lambda$ are linearly dependent only in the case when the determinant

$$\delta = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \lambda_1 & \lambda_2 \end{vmatrix} = 0.$$

As a result we obtain two essentially different representations of algebras $AG_1(1.n)$ and $AG_2(1.n)$ for $\delta = 0$ and $\delta \neq 0$, in contrast to the case of a single diffusion equation (the

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nonlinear diffusion equation invariant with respect of a set of $AG_2(1.n)$ subalgebras was studied in [2, 3]).

Note that in the case when the system (1) is a pair of complex conjugate Schrödinger equations, i.e. $U = \overset{*}{V}, \lambda_1 = \overset{*}{\lambda}_1 = \mathrm{i}$, the operators $I_\alpha$ and $Q_\lambda$ are linearly independent. This results in the fact that nonlinear generalizations of Schrödinger equations, preserving their symmetry [1], differ essentially from nonlinear generalizations of the diffusion system (1) for $\delta = 0$.

Now consider a system of quasilinear generalizations of diffusion equations (1) of the form

$$\begin{aligned} \lambda_1 U_t &= A_{ab}U_{ab} + C_{ab}V_{ab} + B_1 \\ \lambda_1 V_t &= D_{ab}U_{ab} + E_{ab}V_{ab} + B_2 \end{aligned} \tag{3}$$

where $A_{ab}, C_{ab}, D_{ab}, E_{ab}, B_1, B_2$ are arbitrary real or complex differentiable functions of $2n+2$ variables $U, V, U_1, \ldots, U_n, V_1, \ldots, V_n$. The indices $a = 1, \ldots, n$ and $b = 1, \ldots, n$ of functions $U$ and $V$ denote differentiating with respect to $x_a$ and $x_b$.

The system (3) generalizes practically all the known nonlinear systems of first- and second-order evolution equations, describing various processes in physics, chemistry and biology (heat-and-mass transfer, filtration of a two-phase liquid, diffusion in chemical reaction etc) [4–7].

In the case of complex $U = \overset{*}{V}, A_{ab} = \overset{*}{E}_{ab}, C_{ab} = \overset{*}{D}_{ab}, B_1 = \overset{*}{B}_2 = B, \lambda_1 = \overset{*}{\lambda}_2 = \mathrm{i}$ the system (3) is transformed into a pair of complex conjugate equations. We treat them as a class of nonlinear generalizations of Schrödinger equations, namely

$$\mathrm{i}U_t = A_{ab}U_{ab} + \overset{*}{D}_{ab}\overset{*}{U}_{ab} + B \tag{4a}$$

$$-\mathrm{i}\overset{*}{U}_t = \overset{*}{A}_{ab}\overset{*}{U}_{ab} + D_{ab}U_{ab} + \overset{*}{B} \tag{4b}$$

(hereinafter the complex conjugate equations (4$b$) are omitted).

For $A_{ab} = D_{ab} = D_{aa} = 0, a \neq b, A_{aa} = -h$ equation (4$a$) is obviously transformed into a Schrödinger equation with nonlinear potential B:

$$\mathrm{i}U_t + h\Delta U = B. \tag{4'}$$

By choice of the corresponding potential $B = B(U, \overset{*}{U}, U_1 \cdots U_n, \overset{*}{U}_1, \ldots, \overset{*}{U}_n)$ a great variety of Schrödinger equation generalizations, known from the literature (see, e.g., [1, 2, 8, 9, 10]), can be obtained.

In the case of zero potential B a classical Schrödinger equation is obtained

$$\mathrm{i}U_t + h\Delta U = 0 \tag{5}$$

invariant under $AG_2(1.n)$ algebra with the basic operators (2) [11], where

$$Q_\lambda = -\frac{\mathrm{i}}{h}(U\partial_U - \overset{*}{U}\partial_{\overset{*}{U}}) \qquad I_\alpha = \alpha(U\partial_U + \overset{*}{U}\partial_{\overset{*}{U}}). \tag{6}$$

Note that the algebra $AG_2(1.n)$ in the case of the Schrödinger equations is called the Schrödinger algebra [11].

In the present paper all the systems of evolution equations of the form (3), invariant under the chain of algebras $AG(1.n) \subset AG_1(1.n) \subset AG_2(1.n)$, are described. The results obtained are illustrated by the examples of the nonlinear Schrödinger equations, reaction–diffusion equations and Hamilton–Jacobi-type systems.

## 2. Description of systems (3) with Galilean symmetry

The algebra of symmetries for the system of equations (1) contains the Galilei operators $G_a, a = 1, \dots, n$, being a mathematical expression of the Galilei relativistic principle for equations (1). The Galilei operators are also known [3] to be closely related with the fundamental solution of the diffusion equation. We recall that if some system of PDEs is invariant with respect to the Galilei algebra or its extention, then it gives a wide range of possibilities for the construction of multiparametric families of exact solutions [1, 12, 22]. Moveover, the Galilei operators and the projective operator (2*d*) generate non-trivial formulae of multiplication of solutions. These formulae can be used to convert stationary (*time-independent) into non-stationary ones with a different structure.*

In view of this it seems reasonable to search for Galilei-invariant nonlinear generalizations of the system (1) in the class of the system (3).

*Theorem 1.* The system of nonlinear equations (3) is invariant under the Galilei algebra in the represention (2*a*), (2*b*) if and only if it has the form:

$$\begin{aligned} \lambda_1 U_t &= \Delta U + U[A_1 \Delta \ln U + C_1 \Delta \ln V + B_1] \\ &\quad + U[A_2 \omega_a \omega_b (\ln U)_{ab} + C_2 \omega_a \omega_b (\ln V)_{ab}] \\ \lambda_2 V_t &= \Delta V + V[D_1 \Delta \ln U + E_1 \Delta \ln V + B_2] \\ &\quad + V[D_2 \omega_a \omega_b (\ln U)_{ab} + E_2 \omega_a \omega_b (\ln V)_{ab}] \end{aligned} \tag{7}$$

where $(\ln U)_{ab} \equiv \partial^2 \ln U/\partial x_a \partial x_b$, $(\ln V)_{ab} \equiv \partial^2 \ln V/\partial x_a \partial x_b$, $\Delta \ln U \equiv (\ln U)_{11} + \cdots + (\ln U)_{nn}$, $\Delta \ln V \equiv (\ln V)_{11} + \cdots + (\ln V)_{nn}$, $\omega = U^{\lambda_2} V^{-\lambda_1}$, $\omega_a = \partial\omega/\partial x_a \equiv (\lambda_2 U_a/U - \lambda_1 V_a/V)\omega$ and $A_k, B_k, C_k, D_k, E_k$, $k = 1, 2$ are arbitrary functions of absolute invariants of the $AG(1.n)$ algebra $\omega$ and $\theta = \omega_a \omega_a$.

The proof of this and the following theorems is based on the classical Lie scheme, which is realized in [3, 12] for obtaining the Galilei invariant equations. The detailed cumbersome calculations are omitted.

Note that in the case where $\lambda_1 = 0$, i.e. the first equation of system (3) being elliptical, the absolute invariants of the Galilei algebra simplify considerably: $\omega = U, \theta = U_a U_a$.

In the case of systems of the form (3) being $AG_1(1.n)$- and $AG_2(1.n)$-invariant the structure of such systems essentially depends on the determinant $\delta$.

*Theorem 2.* The nonlinear system (3) is invariant with respect to algebra $AG_1(1.n)$ with basic operators (2*a*)–(2*c*) if and only if it has the form:

(i) In the case $\delta \neq 0$

$$\begin{aligned} \lambda_1 U_t &= \Delta U + U[A_1(\hat\theta) \Delta ln U + A_2(\hat\theta) \Delta \ln V + \omega^{-2/\delta} B_1(\hat\theta)] \\ &\quad + U\omega^{2/\delta - 2}[C_1(\hat\theta)\omega_a\omega_b(\ln U)_{ab} + C_2(\hat\theta)\omega_a\omega_b(\ln V)_{ab}] \\ \lambda_2 V_t &= \Delta V + V[D_1(\hat\theta) \Delta \ln U + D_2(\hat\theta) \Delta \ln V + \omega^{-2/\delta} B_2(\hat\theta)] \\ &\quad + V\omega^{2/\delta - 2}[E_1(\hat\theta)\omega_a\omega_b(\ln U)_{ab} + E_2(\hat\theta)\omega_a\omega_b(\ln V)_{ab}]. \end{aligned} \tag{8}$$

(ii) In the case $\delta = 0$

$$\begin{aligned} \lambda_1 U_t &= \Delta U + U[A_1(\omega) \Delta ln V + A_2(\omega) \Delta \ln V + \omega_a \omega_a B_1(\omega)] \\ &\quad + U(\omega_{a_1}\omega_{a_1})^{-1}\omega_a\omega_b[C_1(\omega)(\ln U)_{ab} + C_2(\omega)(\ln V)_{ab}] \\ \lambda_2 V_t &= \Delta V + V[D_1(\omega) \Delta \ln U + D_2(\omega) \Delta Ln V + \omega_a \omega_a B_2(\omega)] \\ &\quad + V(\omega_{a_1}\omega_{a_1})^{-1}\omega_a\omega_b[E_1(\omega)(\ln U)_{ab} + E_2(\omega)(\ln U)_{ab}] \end{aligned} \tag{9}$$

where $A_k, B_k, C_k, D_k, E_k, k = 1, 2$ are arbitrary functions, $\hat{\theta} = \omega_a\omega_a\omega^{2/\delta-2}$ and $\omega$ are the absolute first-order invariants of the algebra $AG_1(1.n)$, $a_1 = 1, \dots, n$ ($\omega_a, \omega$—see theorem 1).

In the case when the first equation of system (3) degenerates into an elliptic ($\lambda_1 = 0$) equation, the absolute invariants in systems (8) and (9) simplify and $\hat{\theta} = U_aU_aU^{2/\alpha_1-2}$ for $\delta \neq 0, \omega = U$ for $\delta = 0$.

*Theorem 3.* The nonlinear system of equations (3) is invariant with respect to algebra $AG_2(1.n)$ with basic operators (2) ($\alpha_1, \alpha_2$ are arbitrary constants) iff it has the form:
(i) In the case $\delta \neq 0$

$$\begin{aligned}\lambda_1 U_t &= \hat{\alpha}_1\Delta U + UA(\hat{\theta})(\lambda_2\Delta\ln U - \lambda_1\Delta\ln V) + U\omega^{-2/\delta}B_1(\hat{\theta})\\ &\quad +(1-\hat{\alpha}_1)U_aU_a/U + U\omega^{2/\delta-2}\omega_a\omega_b[\lambda_2(lnU)_{ab} - \lambda_1(\ln V)_{ab}]C(\hat{\theta})\\ \lambda_2 V_t &= \hat{\alpha}_2\Delta V + VD(\hat{\theta})(\lambda_2\Delta\ln U - \lambda_1\Delta\ln V) + V\omega^{-2/\delta}B_2(\hat{\theta})\\ &\quad +(1-\hat{\alpha}_2)V_aV_a/V + V\omega^{2/\delta-2}\omega_a\omega_b[\lambda_2(lnU)_{ab} - \lambda_1(\ln V)_{ab}]E(\hat{\theta}).\end{aligned} \tag{10}$$

(ii) In the case when $\delta = 0$

$$\begin{aligned}\lambda_1 U_t &= \hat{\alpha}_1\Delta U + UA(\omega)(\lambda_2\Delta\ln U - \lambda_1\Delta\ln V) + U\omega_a\omega_aB_1(\omega)\\ &\quad +(1-\hat{\alpha}_1)U_aU_a/U + U(\omega_{a_1}\omega_{a_1})^{-1}\omega_a\omega_b[\lambda_2(lnU)_{ab} - \lambda_1(\ln V)_{ab}]C(\omega)\\ \lambda_2 V_t &= \hat{\alpha}_2\Delta V + VD(\omega)(\lambda_2\Delta\ln U - \lambda_1\Delta\ln V) + V\omega_a\omega_aB_2(\omega)\\ &\quad +(1-\hat{\alpha}_2)V_aV_a/V + V(\omega_{a_1}\omega_{a_1})^{-1}\omega_a\omega_b[\lambda_2(lnU)_{ab} - \lambda_1(\ln V)_{ab}]E(\omega)\end{aligned} \tag{11}$$

where $A, B_1, B_2, C, D, E$ are arbitrary functions, $\hat{\alpha}_k = -2\alpha_k/n$, $k = 1, 2$ ($\alpha_k$–see the operator $I_\alpha$).

It should be noted that in the case where $\alpha_1\alpha_2 \neq 0$ the systems (10) and (11) can be reduced by the local substitution $U \to U^{\hat{\alpha}_1}$, $V \to V^{\hat{\alpha}_2}$ to systems of the same structure, but with $\hat{\alpha}_k = 1$, i.e. $\alpha_k = -n/2$. The specific case of $\alpha_1 = \alpha_2 = 0$ will be considered in what follows.

The classes of $AG_2(1.n)$-invariant systems (10) and (11) thus obtained contain, in particular, such genaralizations of equations (1) as ($\delta \neq 0$)

$$\lambda_1 U_t = \Delta U + e_1U(\lambda_2\Delta\ln U - \lambda_1\Delta\ln V)$$

$$\lambda_2 U_t = \Delta V + e_2V(\lambda_2\Delta\ln U - \lambda_1\Delta\ln V)$$

and ($\delta = 0$)

$$U_t = \Delta U + e_1U\frac{\partial(UV^{-1})}{\partial x_a}\frac{\partial(UV^{-1})}{\partial x_a}$$

$$V_t = \Delta V + e_2V\frac{\partial(UV^{-1})}{\partial x_a}\frac{\partial(UV^{-1})}{\partial x_a}$$

where $e_1, e_2 \in \mathbb{R}$.

In the case where the first of equations (3) degenerates into an elliptical one ($\lambda_1 = 0$), the $AG_2(1.n)$-invariant systems of equations are simply

$$\begin{aligned} 0 &= A_1(\hat{\theta})\Delta U + A_2(\hat{\theta})(U_{a_1}U_{a_1})^{-1}U_aU_bU_{ab} + U^{1-2/\alpha_1}B_1(\hat{\theta}) \\ &\quad + UC(\hat{\theta})[\Delta \ln V - (U_{a_1}U_{a_1})^{-1}U_aU_b(\ln V)_{ab}] \\ \lambda_2 V_t &= \hat{\alpha}_2\Delta V + \frac{V}{U}D_1(\hat{\theta})\Delta U + \frac{V}{U}D_2(\hat{\theta})(U_{a_1}U_{a_1})^{-1}U_aU_bU_{ab} \\ &\quad + (1-\hat{\alpha}_2)V_aV_a/V + VU^{-2/\alpha_1}B_2(\hat{\theta}) \\ &\quad + VE(\hat{\theta})[\Delta \ln V - (U_{a_1}U_{a_1})^{-1}U_aU_b(\ln V)_{ab}] \end{aligned} \tag{12}$$

if $\delta \neq 0$, and

$$\begin{aligned} 0 &= A_1(U)\Delta U + A_2(U)(U_{a_1}U_{a_1})^{-1}U_aU_bU_{ab} + U_aU_aB_1(U) \\ &\quad + C(U)[\Delta \ln V - (U_{a_1}U_{a_1})^{-1}U_aU_b(\ln V)_{ab}] \\ \lambda_2 V_t &= \hat{\alpha}_2\Delta V + VD_1(U)\Delta U + VD_2(U)(U_{a_1}U_{a_1})^{-1}U_aU_bU_{ab} + VU_aU_aB_2(U) \\ &\quad + (1-\hat{\alpha}_2)V_aV_a/V + VE(U)[\Delta \ln V - (U_{a_1}U_{a_1})^{-1}U_aU_b(\ln V)_{ab}] \end{aligned} \tag{13}$$

if $\delta = 0$. In equations (12), (13) $A_k$, $B_k$, $D_k$, $E$, $C$ are arbitrary functions, $\hat{\theta} = U_aU_aU^{2/\alpha_1-2}$ and $\hat{\alpha}_2 = -2\alpha_2/n$. In [13] integration of two-dimensional systems of equations (12), (13) form was reduced to the integration of linear heat equation with a source.

## 3. Galilei-invariant nonlinear generalizations of the Schrödinger equation

As has been noted above, a class of nonlinear generalization of Schrödinger equation (4) is a specific case of the systems of evolution equations (3). On the basis of theorems 1, 2 and 3 this enables one to describe all quasilinear generalizations of the Schrödinger equation (5), which are invariant with respect to a chain of algebras $AG(1.n) \subset AG_1(1.n) \subset AG_2(1.n)$.

*Corollary 1.* In the class of nonlinear equations of the form (4) the algebra $AG(1.n)$ (2*a*), (2*b*) with $Q_\lambda = -(\mathrm{i}/h)(U\partial_U - \overset{*}{U}\partial_{\overset{*}{U}})$ is admitted only for equations given by

$$\begin{aligned} \mathrm{i}U_t + h\Delta U &= U[A_1\Delta \ln U + A_2\Delta \ln \overset{*}{U} + B] \\ &\quad + U[A_3|U|_a|U|_b(\ln U)_{ab} + A_4|U|_a|U|_b(\ln \overset{*}{U})_{ab}] \end{aligned} \tag{14}$$

where $A_j$, $j = 1, 2, 3, 4$ and $B$ are arbitrary complex functions of two arguments $|U|$ and $|U|_a|U|_a$; $|U|^2 = U\overset{*}{U}$, $|U|_a = \partial|U|/\partial x_a$.

In the case $A_j = 0$ the class of equations (14) is reduced to the equation

$$\mathrm{i}U_t + h\Delta U = UB(|U|, |U|_a|U|_a) \tag{15}$$

obtained in [1, 12], whose specific case is a Schrödinger equation with power nonlinearity $U|U|^\beta$, $\beta =$ constant.

By using the identities

$$\begin{aligned} &\Delta \ln |U|^2 = (\Delta|U|^2 - 4|U|_a|U|_a)/|U|^2 \\ &\mathrm{Re}(\Delta U/U) + |\nabla U|^2/|U|^2 = \Delta \ln |U| + |U|_a|U|_a/|U|^2 \\ &\mathrm{Im}(\Delta U/U - U_aU_a/U^2) = (\Delta \ln U - \Delta \ln \overset{*}{U})/2\mathrm{i} \end{aligned}$$

it is easy to show that the class of the Galilei-invariant equations (14) contains the equation

$$iU_t + \Delta U = \frac{id}{2} U\Delta|U|^2/|U|^2 + U[d_1(\mathrm{Re}(\Delta U/U) + |\nabla U|^2/|U|^2)$$
$$+d_2\,\mathrm{Im}(\Delta U/U - (\nabla U/U)^2) + d_3(\mathrm{Re}(\nabla U/U)^2) + |\nabla U|^2/|U|^2)]$$

where $\nabla U = (\partial U/\partial x_1, \ldots, \partial U/\partial x_n)$, $d_1, d_2, d_3 \in \mathbb{R}$, proposed in [9] from certain physical considerations. By the way, a nonlinear generalization of the Schrödinger equation [8]

$$iU_t = (id_1 - h)\Delta U + id_1 U|\nabla U|^2/|U|^2 + UB(|U|)$$

does not preserve Galilean symmetry of the linear Schrödinger equation. Instead it would be appropriate to propose Galilei-invariant nonlinear equations of the class (14)

$$iU_t = c\Delta U + (h-c)\overset{*}{U}(\nabla U)^2/|U|^2 + UB(|U|)$$

and [13]

$$iU_t = -h\Delta U + cU\Delta|U|^2/|U|^2 + UB(|U|)$$

where $c$ is arbitrary complex constant and $B$ is an arbitrary complex function.

*Corollary 2.* In the class of nonlinear equations of the form (4) the algebra $AG_1(1.n)$ (2*a*), (2*b*), (2*c*), (6) is admitted only for equations given by:
(i) In the case $\alpha \neq 0$

$$iU_t + h\Delta U = U[D_1\Delta \ln U + D_2 \Delta \ln \overset{*}{U} + |U|^{-2/\alpha} B]$$
$$+U|U|^{2/\alpha-2}[D_3|U|_a|U|_b(\ln U)_{ab} + D_4|U|_a|U|_b(\ln \overset{*}{U})_{ab}] \tag{16}$$

where $D_j$, $j = 1, 2, 3, 4$ and $B$ are arbitrary complex functions of the argument $|U|^{2/\alpha-2}|U|_a|U|_a$.
(ii) In the case $\alpha = 0$

$$iU_t + h\Delta U = U[D_1\Delta \ln U + D_2 \Delta \ln \overset{*}{U} + |U|_a|U|_a B]$$
$$+U(|U|_{a_1}|U|_{a_1})^{-1}[D_3|U|_a|U|_b(\ln U)_{ab} + D_4|U|_a|U|_b(\ln \overset{*}{U})_{ab}] \tag{17}$$

where $D_j = D_j(|U|)$, $j = 1, 2, 3, 4$ and $B = B(|U|)$ are arbitrary complex functions.

It is easily seen that the class of the $AG_1(1.n)$-invariant equations (14) contains the well known nonlinear Schrödinger equation

$$iU_t + h\Delta U + cU|U|^2 = 0 \tag{18}$$

which in the case $n = 1$ is integrated by inverse scattering method [14]. Note that in the case $n = 2$ equation (17) is invariant under the $AG_2(1.2)$ algebra [12, 15].

*Corollary 3.* Within the class of nonlinear equations of the form (4) the algebra $AG_2(1.n)$ (2), (6) for $\alpha = -n/2$ of the linear Schrödinger equation (5) is conserved only for equations given by

$$iU_t + h\Delta U = UE_1\Delta ln|U| + U|U|^{4/n}B + U|U|^{-4/n-2}E_2|U|_a|U|_b(ln|U|)_{ab}. \tag{19}$$

In equation (19) $E_1, E_2$ and $B$ are arbitrary complex functions of the argument $|U|^{-4/n-2}|U|_a|U|_a$, which is an absolute invariant of the generalized Galilei algebra $AG_2(1.n)$.

If we consider a representation of the $AG_2(1.n)$ algebra with basic operators (2), (6) for $\alpha = 0$, a principally different class of quasilinear second-order equations, invariant with respect to this algebra, namely

$$iU_t + hU_aU_a/U = UE_1(|U|)\Delta ln|U| + U|U|_a|U|_aB(|U|) \\ +UE_2(|U|)(|U|_{a_1}|U|_{a_1})^{-1}|U|_a|U|_b(ln|U|)_{ab} \qquad (20)$$

is obtained.

It is easily seen that within the class of (20) there is not a single linear equation, the simplest one among them being Hamilton–Jacobi equation for a complex function:

$$iU_t + hU_aU_a/U = 0$$

which is reduced to a standard form

$$iW_t + hW_aW_a = 0 \qquad W_a = \frac{\partial W}{\partial x_a} \qquad W_t = \frac{\partial W}{\partial t}$$

by a local substitution $U = \exp W$, $W = W(t, x_1, \ldots, x_n)$.

In the case $E_1 = E_2 = 0$ the equation

$$iU_t + h\Delta U = U|U|^{4/n}B \qquad (21)$$

is obtained from the class of equations (19), which had been obtained in [1, 12]. Note that for $B = c =$ constant equation (21) is transformed into an equation with fixed power nonlinearity, studied in a series of papers (for $n = 1$ [16, 17], $n = 2$ [18], and $n = 3$ [1, 2, 12, 19]). In [1, 12] multiparametric families of invariant solutions of equation (21) of the form

$$iU_t + h\Delta U = cU\frac{|U|_a|U|_a}{|U|^2}$$

have also been constructed and systematized.

Being written in the case of one spatial variable ($n = 1$), after simple transformations the class of equations (19) is given by

$$iU_t + hU_{xx} = UE_1(ln|U|)_{xx} + U|U|^4B \qquad U = U(t, x) \qquad x = x_1 \qquad (22)$$

$E_1$ and $B$ being arbitrary complex functions of the argument $|U|^{-3}|U|_x$.

Obviously, a specific case of (22) is given by

$$iU_t + hU_{xx} + c_1U|U|^4 + c_2U|U||U|_x = 0 \qquad (23)$$

which for $h = 1, c_1 = 1, c_2 = 4$ is known as the Eckhaus equation [20, 21]. Equation (23) has been studied in detail for arbitrary constant values of $c_1$ and $c_2$ in [22]. A multidimensional generalization of (23), posessing $AG_2(1.n)$ symmetry, can be proposed:

$$iU_t + h\Delta U + c_1U|U|^{4/n} + c_2U|U|^{-1+2/n}(|U|_a|U|_a)^{1/2} = 0. \qquad (24)$$

## 4. Galilei-invariant systems of Hamilton–Jacobi-type

It should be noted that the local substitution $U = M(\hat{U}), V = N(\hat{V})$, where $M, N$ are arbitrary differentiable functions, reduces any system of equations with the symmetry $AG(1.n)$, $AG_1(1.n)$ or$AG_2(1.n)$ to a locally equivalent system with the same symmetry, but with a different representation of the operators $Q_\lambda$ and $I_\alpha$, namely

$$\hat{Q}_\lambda = \lambda_1 M\left(\frac{dM}{d\hat{U}}\right)^{-1}\partial_{\hat{U}} + \lambda_2 N\left(\frac{dN}{d\hat{V}}\right)^{-1}\partial_{\hat{V}}$$

$$I_\alpha = \alpha_1 M\left(\frac{dM}{d\hat{U}}\right)^{-1}\partial_{\hat{U}} + \alpha_2 N\left(\frac{dN}{d\hat{V}}\right)^{-1}\partial_{\hat{V}}.$$

In the particular case where $M = \exp(\hat{U})$, $N = \exp(\hat{V})$, we obtain

$$\hat{Q}_\lambda = \lambda_1 \partial_{\hat{U}} + \lambda_2 \partial_{\hat{V}} \qquad I_\alpha = \alpha_1 \partial_{\hat{U}} + \alpha_2 \partial_{\hat{V}}. \tag{25}$$

In this case the class of equation systems, invariant with respect to $AG_2(1.n)$ algebra in the representation (2), (25), for $\delta = 0$ is given by

$$\begin{aligned}
\lambda_1 \hat{U}_t &= \hat{\alpha}_1 \Delta \hat{U} + A(\hat{\omega})(\lambda_2 \Delta \hat{U} - \lambda_1 \Delta \hat{V}) + \hat{\omega}_a \hat{\omega}_a B_1(\hat{\omega}) \\
&\quad + \hat{U}_a \hat{U}_a + C(\hat{\omega})(\hat{\omega}_{a_1} \hat{\omega}_{a_1})^{-1} \hat{\omega}_a \hat{\omega}_b [\lambda_2 \hat{U}_{ab} - \lambda_1 \hat{V}_{ab}] \\
\lambda_2 \hat{V}_t &= \hat{\alpha}_2 \Delta \hat{V} + D(\hat{\omega})(\lambda_2 \Delta \hat{U} - \lambda_1 \Delta \hat{V}) + \hat{\omega}_a \hat{\omega}_a B_2(\hat{\omega}) \\
&\quad + \hat{V}_a \hat{V}_a + E(\hat{\omega})(\hat{\omega}_{a_1} \hat{\omega}_{a_1})^{-1} \hat{\omega}_a \hat{\omega}_b [\lambda_2 \hat{U}_{ab} - \lambda_1 \hat{V}_{ab}]
\end{aligned} \tag{26}$$

where $\hat{\omega} = \lambda_2 \hat{U} - \lambda_1 \hat{V}$, $\hat{\omega}_a = \lambda_2 \hat{U}_a - \lambda_1 \hat{V}_a$ and $A, B_1, B_2, C, D, E$ are arbitrary differentiable functions.

In the case where $\hat{\alpha}_1 = \hat{\alpha}_2 = 0$, $A = C = D = E = 0$ the system of equations (26) is reduced to the systems of the form (the symbols $\hat{}$ being omitted below)

$$\begin{aligned}
\lambda_1 U_t &= U_a U_a + \omega_a \omega_a B_1(\omega) \\
\lambda_1 V_t &= V_a V_a + \omega_a \omega_a B_2(\omega) \qquad \lambda_1 \lambda_2 \neq 0.
\end{aligned} \tag{27}$$

It is natural to call the system (27) a generalization of the non-coupled system of Hamilton–Jacobi (HJ) equations

$$\begin{aligned}
\lambda_1 U_t &= U_a U_a \\
\lambda_1 V_t &= V_a V_a.
\end{aligned} \tag{28}$$

In contrast to the symmetry of a single HJ equation [2,23], the local symmetry of the system (28) is exhausted by the $AG_2(1.n)$ algebra (2),(25) for $\alpha_1 = \alpha_2 = 0$ with the additional operators

$$P_V = \partial_V \qquad D_1 = -t\partial_t + U\partial_U + V\partial_V. \tag{29}$$

Thus, all the nonlinear generalizations of the form

$$\begin{aligned}
\lambda_1 U_t &= U_a U_a + B_1(U,\ V,\ U_1 \ldots U_n,\ V_1 \ldots V_n) \\
\lambda_1 V_t &= V_a V_a + B_2(U,\ V,\ U_1 \ldots U_n,\ V_1 \ldots V_n)
\end{aligned} \tag{30}$$

of the HJ system, preserving its symmetry $AG_2(1.n)$, are exhaused by the system (27).

Among the nonlinear generalizations of the HJ system (27), a system of equations with unique symmetry properties exists, namely for $B_1 = 0, B_2 = -1/(\lambda_2)^2$ (in what follows $\lambda_1 = 1, \lambda_2 = \lambda$).

*Theorem 4.* The maximal (in the sense of Lie) algebra of the invariance for the system of equations

$$\begin{aligned}
U_t &= U_a U_a \\
V_t &= -\lambda U_a U_a + 2U_a V_a
\end{aligned} \tag{31}$$

is generated by the basic operators

$$\begin{aligned}
&P_t \qquad P_a \qquad J_{ab} \qquad Q_\lambda = \lambda \partial_U - \partial_V \qquad X = W\partial_V \\
&G_a = tP_a - \tfrac{1}{2} x_a Q_\lambda \qquad D = 2tP_t + x_a P_a \\
&\Pi = t^2 P_t + t x_a P_a - \tfrac{1}{4}|x|^2 Q_\lambda \\
&G_a^1 = UP_a - \tfrac{1}{2} x_a P_t \qquad D_1 = 2UP_U + x_a P_a \\
&\Pi_1 = U^2 P_U + U x_a P_a - \tfrac{1}{4}|x|^2 P_t \\
&K_a = x_a t P_t - (2tU + \tfrac{1}{2}|x|^2) P_a + x_a x_b P_b + x_a U Q_\lambda
\end{aligned} \tag{32}$$

where $W$ is an arbitrary differentiable function of $\lambda U - V$.

Note that the presence of the operator $X$ including an arbitrary function $W$ in the invariance algebra for the system (31) is natural, since the second equation of the system is linear with respect to the function $V$. Much more interesting is the fact the system (31) can be considered as a generalization of the classical HJ equation to the case of two unknown functions, since for $W = 1$ the operators (32) generate the same algebra as the HJ equation. We consider this fact to be very important, since a *trivial* generalization of the above-mentioned equation to the system (28) does *not* preserve the symmetry of the HJ equation.

## 5. Galilei-invariant reaction–diffusion systems

We now consider a nonlinear system of evolution equations given by

$$\begin{aligned} \lambda_1 U_t &= \Delta U + f(U, V) \\ \lambda_2 V_t &= \Delta V + g(U, V) \end{aligned} \tag{33}$$

where $f, g$ are arbitrary differentiable functions. The systems of reaction–diffusion equations (33) has been studied intensively of late (see, e.g., [4, 6, 7]). As follows from theorems 1, 2 and 3, the class of systems (33) contains systems with broad symmetry. In particular, all the systems of equations of the form

$$\begin{aligned} \lambda_1 U_t &= \Delta U + U f(\omega) \qquad \omega = U^{\lambda_2} V^{-\lambda_1} \\ \lambda_2 V_t &= \Delta V + V g(\omega) \end{aligned} \tag{34}$$

will be invariant under the Galilei algebra $AG(1.n)$.

*Note 1.* In the case where $\lambda_2 = \lambda_1 = \lambda$, $f = d_1((U+V)/V)^{d_0} - 1$, $g = d_2((U+V)/V)^{d_0} - d_3$ and $d_0, d_1, d_2, d_3 \in \mathbb{R}$ the system (34) is the particular case of the conservation equations for normal and mutant cells [7, 24].

In the case where $f = \beta_1 \omega^{-2/\delta}$, $g = \beta_2 \omega^{-2/\delta}$, $\delta \neq 0$ ($\delta$ is defined in the introduction) there will be invariance under the algebra $AG_1(1.n)$. Finally, for $\delta = -n(\lambda_2 - \lambda_1)/2$, i.e. $\alpha_1 = \alpha_2 = -n/2$, the system of equations

$$\begin{aligned} \lambda_1 U_t &= \Delta U + \beta_1 U^{1+\lambda_2\gamma} V^{-\lambda_1\gamma} \\ \lambda_2 V_t &= \Delta V + \beta_2 V^{1-\lambda_1\gamma} U^{\lambda_2\gamma} \end{aligned} \tag{35}$$

is obtained (where $\gamma = 4/(n(\lambda_2 - \lambda_1))$, $\lambda_2 \neq \lambda_1$, $\beta_k \in \mathbb{R}$), preserving the $AG_2(1.n)$-symmetry of the linear system (1).

*Note 2.* For $\lambda_2 = -\lambda_1 = \lambda$ the diffusion system (33) is reduced by the substitution

$$U = Y + Z \qquad V = Y - Z \qquad Y = Y(t, x) \qquad Z = Z(t, x) \tag{36}$$

to the system of equations

$$\begin{aligned} -\lambda Y_t &= \Delta Z + f_1(Y, Z) \\ \lambda Z_t &= \Delta Y + g_1(Y, Z) \end{aligned}$$

whose invariance under the chain of algebras $AG(1.n) \subset AG_1(1.n) \subset AG_2(1.n)$ with the unit operator $Q_\lambda = \lambda Y \partial_Z + \lambda Z \partial_Y$ is described by the substitution (36) being applied to the system of equations of the form (33) with the corresponding symmetry.

It is interesting to consider the system (33) in the case where one of the equations degenerates into an elliptical one. Without reducing generality we consider $\lambda_2 = 0$, $\lambda_1 = 1$.

Then according to theorem 1, all systems of the form (33) for $\lambda_2 = 0, \lambda_1 = 1$ and posessing $AG(1.n)$ symmetry are given by

$$U_t = \Delta U + Uf(V) \tag{37a}$$

$$0 = \Delta V + g(V) \tag{37b}$$

where $f$ and $g$ are arbitrary functions.

For the system (37) a clear physical treatment can be suggested. Namely, equation (37$a$) is the heat equation with a spatial source of energy absorption (extraction) $q = Uf(V)$, proportional to the temperature $U$, with an additional constraint of the elliptical equation (37$b$) being imposed on the proportionality coeficient $f(V)$ (in particular we can consider $f(V) = V$). Thus we have obtained a class of nonlinear heat equations with an additional constraint for the source that preserve the Galilean symmetry of the linear heat equation. This result is quite non-trivial, since it is a well known fact that among nonlinear heat equations with a source

$$U_t = \Delta U + q(U)$$

not a single one is invariant with respect to the Galilei algebra $AG(1.n)$ [3]. As can be seen, this 'symmetry contradiction' between the linear and nonlinear heat equations can be solved in two ways: either the source is supposed to depend explicitly on temperature and the independent variables $t, x_1, \ldots, x_n$ [3], or an additional constraint equation (37$b$) is imposed upon the source as above.

It should be noted that in the case $f = \beta_1 V^{2/\alpha_2}$, $g = \beta_2 V^{1+2/\alpha_2}$, $0 \neq \alpha_2$, $\beta_k \in \mathbb{R}$, the system (37) is invariant under the $AG_1(1.n)$ algebra (2$a$)–(2$c$). If the system (37) has the form

$$U_t = \Delta U + \beta_1 U V^{4/n} \tag{38a}$$

$$0 = \Delta V + \beta_2 V^{1+4/n} \tag{38b}$$

it is invariant under the $AG_2(1.n)$ algebra with basic operators (2) for $\lambda_2 = 0$, $\lambda_1 = 1$, i.e. the heat equation (38$a$) with nonlinear constraint (38$b$) for the source conserves all the non-trivial Lie symmetry of the linear heat equation

$$U_t = \Delta U.$$

*Note 3.* If $V$ is a fixed given function on independent variables $t, x_1, \ldots, x_n$, equation (38$a$) can lose any symmetry.

In conclusion, the interesting system of the form (33) should be considered, namely

$$\begin{aligned} \lambda U_t &= \Delta U + \beta_1 U^2 V^{-1} \\ \lambda V_t &= \Delta V + \beta_2 U \qquad \beta_1 \neq \beta_2. \end{aligned} \tag{39}$$

*Theorem 5.* The maximal algebra of invariance for the system (39) is the generalized Galilei algebra with the basic operators (2$a$), (2$b$) and

$$D = 2tP_t + x_a P_a - 2U\partial_U - \left(\frac{n}{2} + \frac{\beta_2}{\beta_1 - \beta_2}\right) Q_\lambda$$

$$\Pi = -t^2 P_t + tD - \frac{1}{4}|x|^2 Q_\lambda - \frac{\lambda}{\beta_1 - \beta_2} V\partial_U.$$

By the way, among the systems of the form (33), in the case where $\lambda_2 = \lambda_1 = \lambda$ *there is not an $AG_2(1.n)$-invariant system in the standard representation* (2). Note that the system (39) can be considered as a particular case of the conservation equations for normal and mutant cells [7, 24].

Some classes of exact solutions for the system (39) have been obtained in [25].

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